

# Nonparametric Estimation of Expected Shortfall

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**ABSTRACT** The expected value of a function of a random variable is estimated by a nonparametric method. The method is based on the use of a kernel function and a bandwidth parameter. The method is applied to the estimation of the expected shortfall of a random variable. The method is compared with the method of the Monte Carlo simulation. The method is shown to be more efficient than the method of the Monte Carlo simulation. The method is applied to the estimation of the expected shortfall of a random variable. The method is compared with the method of the Monte Carlo simulation. The method is shown to be more efficient than the method of the Monte Carlo simulation. The method is applied to the estimation of the expected shortfall of a random variable. The method is compared with the method of the Monte Carlo simulation. The method is shown to be more efficient than the method of the Monte Carlo simulation.



nonp e c e ne e o nd pp e fo en y n y n e con e of  
po fo o oc on

An d n e of e nonp e c e od e n ode f ee ndence ode  
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e od c n p y n c n o e Anode d n e of e nonp e c pp o  
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e on fo o n dependen e e nd M on nd dependen G en nd

In order to study the asymptotic behavior of the proposed estimator of the hazard function, we first consider the asymptotic behavior of the empirical cumulative distribution function. Let  $F_n(t) = \frac{1}{n} \sum_{t=1}^n I_{Y_t \leq t}$  be the empirical cumulative distribution function based on the sample  $\{Y_t\}_{t=1}^n$ . Then, by the central limit theorem, we have

## 2. NONPARAMETRIC ESTIMATORS

A nonparametric estimator of the hazard function is proposed by Chen and Tang (2005) as follows:

$$\hat{h}_p = \frac{\sum_{t=1}^n Y_t I_{Y_t \geq \hat{\nu}_p}}{\sum_{t=1}^n I_{Y_t \geq \hat{\nu}_p}} \quad np^{-1} \sum_{t=1}^n Y_t I_{Y_t \geq \hat{\nu}_p}$$

where  $I_{\cdot}$  is the indicator function.

The kernel estimator of the hazard function is proposed by Chen and Tang (2005) as follows:

$$S_h(z) = n^{-1} \sum_{t=1}^n G_h(z - Y_t)$$

where  $G_h$  is the kernel function. The kernel estimator of the hazard function is proposed by Chen and Tang (2005) as follows:

$$\hat{h}_{p,h} = np^{-1} \sum_{t=1}^n Y_t G_h(\hat{\nu}_{p,h} - Y_t)$$

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<sup>2</sup>Its statistical properties and how to obtain the standard errors are considered in Chen and Tang (2005). See also Cai (2002) and Fan and Gu (2003) for kernel estimators.

Based on the properties of the sequence  $\nu_{p,h}$  or  $\nu_p$ , we proceed to the next step of the proof. The proof is based on the cyclicity of the non-oriented edges of  $\mathcal{F}_p$ . Condition (1) is satisfied for the focus of the next section.

### 3. MAIN RESULTS

The purpose of this section is to prove the convergence of the sequence  $\alpha_k$  under the conditions (1) and (2).

Let  $\mathcal{F}_k^l$  be the  $\sigma$ -algebra generated by  $\{Y_t, k \leq t \leq l\}$  for  $l > k$ . The  $\alpha_k$  is a decreasing sequence of non-negative numbers.

$$\alpha_k = \sup_{A \in \mathcal{F}_1^k, B \in \mathcal{F}_{i+k}^\infty} |P(AB) - P(A)P(B)|.$$

The sequence  $\alpha_k$  is defined as the dependence coefficient of  $\alpha_k$  in the limit  $k \rightarrow \infty$ . The dependence coefficient is defined by the maximum of the absolute value of the difference between the joint probability and the product of the marginal probabilities. The dependence coefficient is defined as the maximum of the absolute value of the difference between the joint probability and the product of the marginal probabilities.

Let  $\rho \in (0, 1)$ ,  $\alpha_k \leq C\rho^k$  for  $k \geq 1$  and suppose that  $C$  is a constant. The dependence coefficient  $\alpha_k$  is defined as the maximum of the absolute value of the difference between the joint probability and the product of the marginal probabilities. The dependence coefficient is defined as the maximum of the absolute value of the difference between the joint probability and the product of the marginal probabilities.

$$K = \int_{-1}^1 y^2 K(y) dy, \quad \sigma_K^2 > 0, \quad \text{and } K(y) \text{ is bounded and Lipschitz continuous.}$$

$$\alpha_k \rightarrow 0 \text{ as } n^{3-\beta} \rightarrow \infty \text{ for any } \beta > 0 \text{ and } n^4 o^2 n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Condition (1) is satisfied for the sequence  $\alpha_k$  in the limit  $k \rightarrow \infty$ . The dependence coefficient is defined as the maximum of the absolute value of the difference between the joint probability and the product of the marginal probabilities. The dependence coefficient is defined as the maximum of the absolute value of the difference between the joint probability and the product of the marginal probabilities.



Under the following conditions, as  $n \rightarrow \infty$

$$\sqrt{pn}\sigma_0^{-1} p d n \hat{\mu}_{p,h} - \mu_p \xrightarrow{d} N(0, \sigma_0^2)$$

and for the

$$Bias \hat{\mu}_{p,h} = -\frac{1}{2} p^{-1} \sigma_K^2 \nu_p \quad \text{and}$$

$$Var \hat{\mu}_{p,h} = p^{-1} n^{-1} \sigma_0^2 p d n \quad \text{or } n^{-1}.$$

By comparing the above conditions with the conditions of the asymptotic normality of the maximum likelihood estimator, we can see that the conditions are satisfied if the following conditions are met: (1) the estimator is unbiased; (2) the estimator is consistent; (3) the estimator is efficient; (4) the estimator is asymptotically normal. The above conditions are satisfied if the following conditions are met: (1) the estimator is unbiased; (2) the estimator is consistent; (3) the estimator is efficient; (4) the estimator is asymptotically normal.

The second part of the above conditions is satisfied if the following conditions are met: (1) the estimator is unbiased; (2) the estimator is consistent; (3) the estimator is efficient; (4) the estimator is asymptotically normal. The above conditions are satisfied if the following conditions are met: (1) the estimator is unbiased; (2) the estimator is consistent; (3) the estimator is efficient; (4) the estimator is asymptotically normal.

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#### 4. SIMULATION STUDY

In this section, we will simulate the above conditions and compare the results with the theoretical results. The simulation results show that the estimator is unbiased, consistent, efficient, and asymptotically normal.

The code below shows the number of observations on the

n

The new  $\rho$  is a function of the  $\rho$  and  $\rho$  density of

$p \leq n/q$  and  $\nu^2 q \leq C p$ . Apply the inequality  $E \left( \sum_{l=[jp]+1}^{(j+1)p} X_l \right)^2 \leq C p$  for  $\alpha \leq n$  and  $\nu^2 q \leq C p$ .

$$\begin{aligned}
 & P\{|F_n - \nu_p - \epsilon_n| > C_1 \epsilon_n\} \\
 & \leq P\left\{-\frac{C_1^2 \epsilon_n^2 q}{\sigma^2 q}\right\} \leq \left\{\frac{C_1^2 \epsilon_n^2 q}{\sigma^2 q}\right\}^{1/2} q \alpha \{n/q\} \quad A
 \end{aligned}$$

$$\begin{aligned}
 & P\{|F_n - \nu_p - \epsilon_n| > C_2 \epsilon_n\} \\
 & \leq P\left\{-\frac{C_2^2 \epsilon_n^2 q}{\sigma^2 q}\right\} \leq P\{-C_2 \epsilon_n q\} \quad A
 \end{aligned}$$

Since  $C_2 > n \epsilon_n^2 \rightarrow \infty$  and  $n q \epsilon_n \rightarrow \infty$ , the above inequality holds for  $n$  large enough. On the other hand, of A, we have  $\alpha \leq n$  and  $p \leq n/q$ .

$$\left\{\left(\frac{C_1^2 \epsilon_n^2 q}{\sigma^2 q}\right)^{1/2} q \alpha \{n/q\}\right\} \leq C \epsilon_n^{-1/2} q \rho^{[n^{1/2} \log^{-1}(n)/2]} \quad A$$

The above inequality holds for  $n$  large enough.  $\square$

Lemma 1. Under the conditions above, for any  $\kappa > 0$ ,

$$n^{-1} \sum Y_t - \nu_p \{I_{Y_t \geq \hat{\nu}_p} - I_{Y_t \geq \nu_p}\} = o_p(n^{-3/4+\kappa}).$$

Proof. Let  $W_t = Y_t - \nu_p \{I_{Y_t \geq \hat{\nu}_p} - I_{Y_t \geq \nu_p}\}$ . Then  $E W_t = -I_{t1} - I_{t2}$ .

$$I_{t1} = E\{Y_t - \nu_p \mid \nu_p \leq Y_t < \hat{\nu}_p \mid \hat{\nu}_p > \nu_p\} \quad \text{and}$$

$$I_{t2} = E\{Y_t - \nu_p \mid \hat{\nu}_p \leq Y_t < \nu_p \mid \hat{\nu}_p < \nu_p\}.$$

For  $a \in \mathbb{R}$ ,  $a > 0$  and  $\eta > 0$ ,

$$I_{t11} = E\{Y_t - \nu_p \mid \nu_p \leq Y_t < \hat{\nu}_p \mid \hat{\nu}_p \geq \nu_p - n^{-a} \eta\},$$

$$I_{t12} = E\{Y_t - \nu_p \mid \nu_p \leq Y_t < \hat{\nu}_p \mid \nu_p < \hat{\nu}_p < \nu_p - n^{-a} \eta\},$$

$$I_{t21} = E\{Y_t - \nu_p \mid \nu_p > Y_t \geq \hat{\nu}_p \mid \hat{\nu}_p \leq \nu_p - n^{-a} \eta\} \quad \text{and}$$

$$I_{t22} = E\{Y_t - \nu_p \mid \nu_p > Y_t \geq \hat{\nu}_p \mid \nu_p > \hat{\nu}_p > \nu_p - n^{-a} \eta\}.$$

Apply the Cauchy-Schwarz inequality for  $k$  and

$$|I_{tk1}| \leq \sqrt{E \hat{\nu}_p - \nu_p^2 P\{|\hat{\nu}_p - \nu_p| \geq n^{-a} \eta\}}.$$

En Le e f c  $E \hat{\nu}_p - \nu_p^2 = O(n^{-1})$

$I_{tk1} \rightarrow$  e ponen y f . A

o e e  $I_{t12}$  è no e  $|I_{t12}| \leq E\{Y_t - \nu_p \mid \nu_p \leq Y_t < \nu_p\} = O(n^{-a})$

$$I_{t12} \leq \int_{\nu_p}^{\nu_p + n^{-a}\eta} dz (z - \nu_p) f(z) = O(n^{-2a})$$

in e e c y e pp o è c n  $I_{t22} = O(n^{-2a})$  è e e nd A  
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$E W_t = O(n^{-1+\kappa})$  A

fo n y p o e  $\kappa$  n n p e

$$E\left[n^{-1} \sum (Y_t - \nu_p) \{I_{Y_t \geq \hat{\nu}_p} - I_{Y_t \geq \nu_p}\}\right] = O(n^{-1+\kappa})$$
 A

è no con de  $Var W_i = O(a)$  /

$$\begin{aligned} E W_t^2 &= E\left[ Y_t - \nu_p^2 \{I_{Y_t \geq \hat{\nu}_p} - I_{Y_t \geq \nu_p}\} \mid I_{Y_t \geq \nu_p} \right] \\ &= E\left[ Y_t - \nu_p^2 \{I_{\nu_p > Y_t \geq \hat{\nu}_p} \mid I_{\hat{\nu}_p > Y_t \geq \nu_p}\} \right] \\ &= E\left[ Y_t - \nu_p^2 I_{\hat{\nu}_p \leq Y_t < \nu_p} \{I_{\hat{\nu}_p \geq \nu_p - n^{-a}\eta} \mid I_{\hat{\nu}_p < \nu_p - n^{-a}\eta}\} \right] \\ &= E\left[ Y_t - \nu_p^2 I_{\hat{\nu}_p > Y_t \geq \nu_p} \{I_{\hat{\nu}_p \geq \nu_p - n^{-a}\eta} \mid I_{\hat{\nu}_p < \nu_p - n^{-a}\eta}\} \right]. \end{aligned}$$

No e

$$E\{I_{\hat{\nu}_p \leq Y_t < \nu_p} \mid I_{\hat{\nu}_p \leq \nu_p - n^{-a}\eta}\} \leq P\{|\hat{\nu}_p - \nu_p| \geq n^{-a}\eta\}$$
 nd

$$E\{I_{\hat{\nu}_p > Y_t \geq \nu_p} \mid I_{\hat{\nu}_p > \nu_p - n^{-a}\eta}\} \leq P\{|\hat{\nu}_p - \nu_p| \geq n^{-a}\eta\}$$

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$$E\{Y_t - \nu_p^2 I_{\hat{\nu}_p \leq Y_t < \nu_p} \mid I_{\hat{\nu}_p \leq \nu_p - n^{-a}\eta}\} = \text{nd}$$

$$E\{Y_t - \nu_p^2 I_{\hat{\nu}_p > Y_t \geq \nu_p} \mid I_{\hat{\nu}_p \geq \nu_p - n^{-a}\eta}\}$$

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 A è e

$$E\{Y_t - \nu_p^2 I \dot{\nu}_p \leq Y_t < \nu_p I \dot{\nu}_p \geq \nu_p - n^{-a}\eta\} = O(n^{-3a}) \quad \text{nd}$$

$$E\{Y_t - \nu_p^2 I \dot{\nu}_p > Y_t \geq \nu_p I \dot{\nu}_p < \nu_p - n^{-a}\eta\} = O(n^{-3a}) .$$

n è e  $E W_t^2 = o(n^{-3/2+\kappa})$  nd A e n  $Var W_t = o(n^{-3/2+\kappa})$   
 By y od fy n e o e de on fo  $Var W_t$  y e o n fo ny  $t_1, t_2$   
 $Cov W_{t_1}, W_{t_2} = o(n^{-3/2+\kappa})$  e efo e

$$Var\left[n^{-1} \sum_{i=1}^n Y_t - \nu_p \{I Y_t \geq \dot{\nu}_p - I Y_t \geq \nu_p\}\right] = o(n^{-3/2+\kappa}) . \quad A$$

o e e A e d y e e e

L MMA Le  $\dot{\beta} = np^{-1} \sum Y_t G_h \nu_p - Y_t$  nd  $\dot{\eta} = n^{-1} \sum_{i=1}^n Y_t K_h \nu_p - Y_t$  nd e  
 cond on

$$Cov\left[\dot{\beta}, \{p - S_h \nu_p\} \{\dot{f} \nu_p - f \nu_p\}\right] = o(n^{-1}) ,$$

$$Cov\left[\dot{\beta}, \dot{\eta} - \eta \{p - S_h \nu_p\}\right] = o(n^{-1}) ,$$

$$Cov\left[\{p - S_h \nu_p\}, \dot{\eta} - \eta \{p - S_h \nu_p\}\right] = o(n^{-1}) .$$

OO è on y p e en e p oof of e p oof fo e o e e e De ne  
 $\beta = E \dot{\beta}$  Le  $\dot{\beta} - \beta = n^{-1} \sum \psi_1 Y_t \dot{f} \nu_p - f \nu_p - n^{-1} \sum \psi_2 Y_t = O_p(2)$  nd  $p - \dot{F}_h \nu_p$   
 $n^{-1} \sum \psi_3 Y_t = O_p(2)$  fo e f nc on  $\psi_j, j = 1, 2, 3$ , nd  $E\{\psi_j Y_t\} = 0$   
 n ce  $\psi_2 Y_t = K_h \nu_p - Y_t - E\{K_h \nu_p - Y_t\}$  nd  $\psi_3 Y_t = G_h \nu_p - Y_t - E\{G_h \nu_p - Y_t\}$   
 n e pp o n B n ey p

$$A \quad |E\left[\dot{\beta} - \beta \{p - S_h \nu_p\} \{\dot{f} \nu_p - f \nu_p\}\right]|$$

$$\leq n^{-2} \sum_{i \geq 1, j \geq 1, i+j \leq n} |E\{\psi_1 Y_i \psi_2 Y_i \psi_3 Y_{i+j}\}| = O(n^{-1-4} n^{-2-2}) \quad A$$

e e nd c e e e d e en pe on on e e nd ce Le p  $\delta$   
 $q = \delta$  nd  $s^{-1} = p^{-1} - q^{-1}$  fo e po e  $\delta$  o e D ydo neq y

$$|E\{\psi_1 Y_i \psi_2 Y_i \psi_3 Y_{i+j}\}| \leq \|\psi_1 Z_1\|_p \|\psi_2 Y_i\|_q \|\psi_3 Z_{i+j}\|_q \alpha^{1/s} .$$



veerpoyn de o

By choosing  $r = n^a$  for  $a \in \mathbb{R}$ , and  $k' = n^c$  for  $c \in \mathbb{R}$ ,  $-a < c$ , we can show the following lemma of A concerning  $n \rightarrow \infty$  hence

$$n \xrightarrow{p} \text{ as } n \rightarrow \infty. \quad \text{A}$$

The following  $S_{n,1} = n^{-1/2} \sum_{j=1}^r W_{j,n} = o_p$

By applying the inequality  $\| \cdot \|_2$  in the following and the condition of  $W_{j,n}$

Le  $\eta = E\{p^{-1}Y_t K_h(Y_t - \nu_p)\} = p^{-1} \int \nu_p - y K(y) f(\nu_p - y) dy = p^{-1} \nu_p f(\nu_p) = O(p^{-2})$   
 et  $Cov\{np^{-1} \sum Y_t K_h(\nu_p - Y_t), \hat{\nu}_{p,h} - \nu_p\} = O(n^{-1})$  d'après le lemme A.

$$E\{np^{-1} \sum Y_t K_h(\nu_p - Y_t) (\hat{\nu}_{p,h} - \nu_p)\} = \eta E(\hat{\nu}_{p,h} - \nu_p) = O(n^{-1}) \\ = -\frac{1}{2} p^{-1} \nu_p f'(\nu_p) \nu_p^2 \sigma_K^2 = o(p^{-3}) = O(n^{-1}).$$

Comme A, A et A,

$$E(\hat{\nu}_{p,h} - \nu_p) = -\frac{1}{2} p^{-1} \sigma_K^2 f''(\nu_p) \nu_p^2 = o(p^{-2}) = O(n^{-1})$$

et  $E(\hat{\nu}_{p,h} - \nu_p)^2 = \sigma_{\hat{\nu}_{p,h}}^2 = o(p^{-2}) = O(n^{-1})$

Le  $A_1 = np^{-1} \sum_{t=1}^n \{Y_t G_h(\nu_p - Y_t) - Y_t K_h(\nu_p - Y_t) (\hat{\nu}_{p,h} - \nu_p)\}$  est une somme de variables indépendantes. Le lemme A et le lemme A donnent

$$Var(A_1) = Var\{np^{-1} \sum Y_t G_h(\nu_p - Y_t)\} + Var\{\hat{\eta}(\hat{\nu}_{p,h} - \nu_p)\} \\ - Cov\{np^{-1} \sum Y_t G_h(\nu_p - Y_t), \hat{\eta}(\hat{\nu}_{p,h} - \nu_p)\}. \quad A$$

et

$$Var\{np^{-1} \sum Y_t G_h(\nu_p - Y_t)\} \\ = n^{-1} p^{-2} \left[ Var\{Y_1 G_h(\nu_p - Y_1)\} + \sum_{k=1}^{n-1} -k/n Cov\{Y_1 G_h(\nu_p - Y_1), Y_{k+1} G_h(\nu_p - Y_{k+1})\} \right].$$

Le  $c_K = \int_{-\infty}^{\infty} y K(y) dy = \int_{-\infty}^{\infty} K(y) y dy = 0$  car  $K$  est une fonction paire.

$$Var\{Y_t G_h(\nu_p - Y_t)\} = \int z^2 G_h^2(\nu_p - z) f(z) dz - p^2 \nu_p^2 = O(p^{-2}) \\ = \int_{-\infty}^{\infty} K(y) dy \left[ \int_{-\infty}^u K(y) dy \left\{ \int_{\nu_p}^{\infty} z^2 f(z) dz + \int_{\nu_p - hu}^{\nu_p} z^2 f(z) dz \right\} \right. \\ \left. - \int_u^{\infty} K(y) dy \left\{ \int_{\nu_p}^{\infty} z^2 f(z) dz + \int_{\nu_p - hu}^{\nu_p} z^2 f(z) dz \right\} \right] - p^2 \nu_p^2 = O(p^{-2}) \\ = Var\{Y_t I_{Y_t \geq \nu_p}\} - \nu_p^2 f(\nu_p) c_K = O(p^{-2}). \quad A$$

q on A et le lemme A,

$$Var\{np^{-1} \sum Y_t G_h(\nu_p - Y_t)\} = p^{-2} Var\{\phi_1(\nu_p)\} = n^{-1} \nu_p^2 f(\nu_p) c_K = o(n^{-1}). \quad A$$

the second order term of the expansion of  $A$

$$\begin{aligned} & \text{Var}\{\eta \hat{\nu}_{p,h} - \nu_p\} = \hat{\eta} - \eta \hat{\nu}_{p,h} - \nu_p \\ & \eta^2 \text{Var}\{\hat{\nu}_{p,h}\} + \eta \text{Cov}\{\hat{\nu}_{p,h}, \hat{\eta} - \eta \hat{\nu}_{p,h} - \nu_p\} + \text{Var}\{\hat{\eta} - \eta \hat{\nu}_{p,h} - \nu_p\}. \end{aligned}$$

By the central limit theorem,  $\eta \hat{\nu}_{p,h} - \nu_p = O_p(n^{-1/2})$

$$\eta^2 \text{Var}\{\hat{\nu}_{p,h}\} = p^{-2} \nu_p^2 \text{Var}\left\{n^{-1} \sum_{t=1}^n I(Y_t > \nu_p)\right\} - p^{-2} n^{-1} b \nu_p^2 f(\nu_p) c_K = o(n^{-1}). \quad A$$

By the law of large numbers,  $E\{\hat{\eta} - \eta\} = o(n^{-1})$  and  $E\{\hat{\nu}_{p,h} - \nu_p\} = o(n^{-1})$ .

$$E\{\hat{\nu}_{p,h} - \nu_p\}^4 \leq Cn^{-2} \quad \text{and} \quad E\{\hat{\eta} - \eta\}^4 = O(n^{-2-3}).$$

Apply the Cauchy-Schwarz inequality and Lemma 4.1

$$\text{Var}\{\hat{\eta} - \eta \hat{\nu}_{p,h} - \nu_p\} = O(n^{-2-3/2}) = o(n^{-1}) \quad \text{and} \quad A$$

$$\text{Cov}\{\eta \hat{\nu}_{p,h} - \nu_p, p^{-1} \hat{\eta} - \eta \hat{\nu}_{p,h} - \nu_p\} = o(n^{-1}). \quad A$$

Combine A, A and A

$$\text{Var}\{\hat{\eta} \hat{\nu}_{p,h} - \nu_p\} = p^{-2} \nu_p^2 \text{Var}\{\phi_2(\nu_p)\} - p^{-2} n^{-1} \nu_p^2 f(\nu_p) c_K = o(n^{-1}). \quad A$$

By Lemma 4.1, the covariance term of the expansion of  $A$

$$\begin{aligned} & \text{Cov}\left\{np^{-1} \sum_{t=1}^n Y_t G_h(\nu_p) - Y_t, \hat{\eta} \hat{\nu}_{p,h} - \nu_p\right\} \\ & \text{Cov}\left\{np^{-1} \sum_{t=1}^n Y_t G_h(\nu_p) - Y_t, \eta f^{-1}(\nu_p) n^{-1} \sum_{t=1}^n G_h(\nu_p) - Y_t\right\} = o(n^{-1}) \\ & np^{-2} \nu_p \left[ \text{Cov}\{Y_t G_h(\nu_p) - Y_t, G_h(\nu_p) - Y_t\} \right. \\ & \left. \sum_{k=1}^{n-1} -k/n \text{Cov}\{Y_1 G_h(\nu_p) - Y_1, G_h(\nu_p) - Y_{k+1}\} \right] = o(n^{-1}) \end{aligned}$$

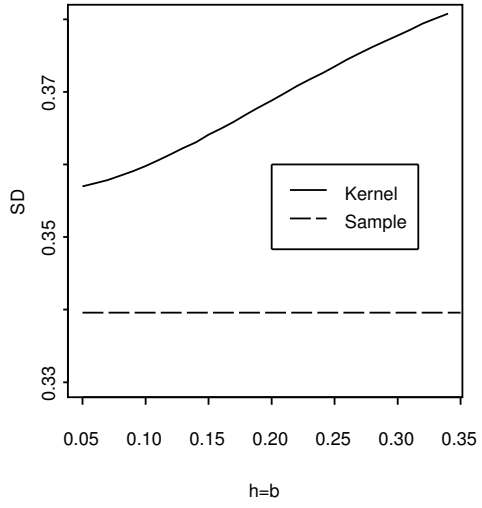
hence  $\text{Cov}\{Y_t G_h(\nu_p) - Y_t, G_h(\nu_p) - Y_t\} = p^{-1} \nu_p f(\nu_p) c_K = o(n^{-1})$

$$\begin{aligned} & \text{Cov}\left\{np^{-1} \sum_{t=1}^n Y_t G_h(\nu_p) - Y_t, np^{-1} \sum_{i=1}^n Y_i K_h(\nu_p) - Y_t \hat{\nu}_{p,h} - \nu_p\right\} = A \\ & n^{-1} p^{-2} \nu_p \text{Cov}\{\phi_1(\nu_p), \phi_2(\nu_p)\} - n^{-1} p^{-2} \nu_p^2 f(\nu_p) c_K = o(n^{-1}). \end{aligned}$$

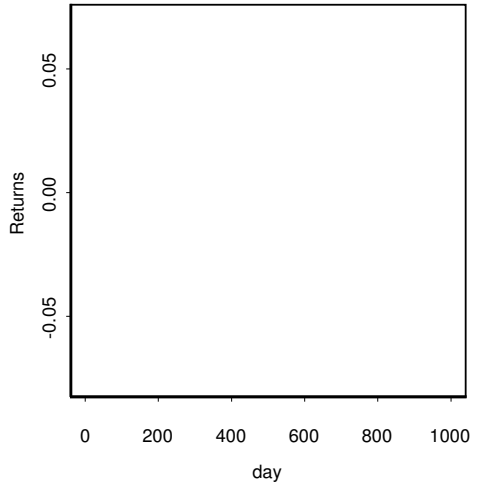


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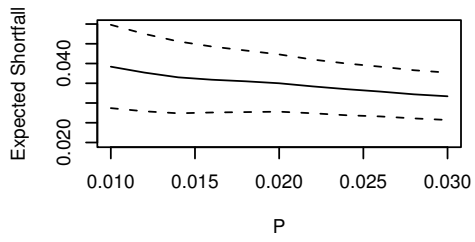
(a) ES Estimates



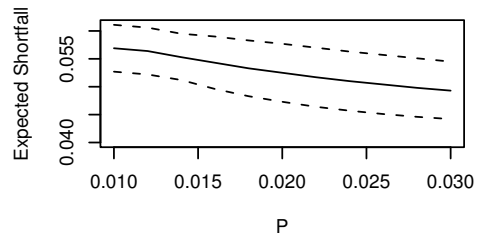




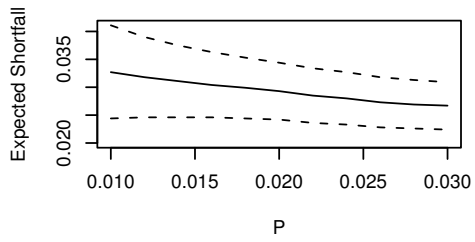
DO ON  $\lambda^2$   $1^2$



CAC  $\lambda^2$   $1^2$



DO ON  $\lambda^2$   $2^2$



*T* Esti tes for  $\nu_{0.01}$   $\mu_{0.01}$  and Standard Errors S.E

Ye	CAC		Do one	
	$\hat{\nu}_{0.01}$	$\hat{\mu}_p$	$\hat{\nu}_{0.01}$	$\hat{\mu}_p$
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